Vol. 13 (2020) *Acta Physica Polonica B Proceedings Supplement* No 4

CAUSAL DYNAMICAL TRIANGULATIONS ON A TORUS*

Jakub Gizbert-Studnicki

Institute of Theoretical Physics, Jagiellonian University
Łojasiewicza 11, 30-348 Kraków, Poland
jakub.gizbert-studnicki@uj.edu.pl

(Received February 18, 2020)

Causal Dynamical Triangulations (CDT) is a non-perturbative lattice approach to quantum gravity where one assumes space-time foliation into spatial hyper-surfaces of fixed topology. Most of the previous studies of CDT were done for the fixed spatial topology of the 3-sphere. We present recent results for the fixed spatial topology of the 3-torus. We argue that the topology change does neither affect the phase structure nor the order of the phase transitions. Thus, the CDT properties seem to be universal, independently of the spatial topology choice.

DOI:10.5506/APhysPolBSupp.13.739

1. Introduction

For nearly 100 years combining general relativity (GR) and quantum field theory (QFT) into a single theory of *quantum gravity* (QG) has been an unsolved problem of theoretical physics. Standard perturbative approaches applied to quantizing GR turned out to be non-renormalizable [1], however, following Weinberg's *asymptotic safety* conjecture [2], it is still possible to formulate a consistent and predictive theory of QG. Asymptotic safety requires that gravitational renormalization group flow equations lead to non-Gaussian ultraviolet (UV) fixed point(s), where QG becomes scale-invariant and can be investigated non-perturbatively. One of possible solutions is thus to use non-perturbative lattice QFT techniques. In the lattice formulation, the UV limit should be associated with a second (or higher) order phase transition. One should also be able to observe the infrared (IR) limit consistent with GR. Thus, investigating the phase structure and the order of the phase transitions plays an important role in lattice QG.

* Presented at the 45^{th} Congress of Polish Physicists, Kraków, September 13–18, 2019.

2. Causal Dynamical Triangulations (CDT)

Among many candidate theories of quantum gravity, Causal Dynamical Triangulations (CDT) is a background-independent, non-perturbative lattice approach based on Feynman's path integral formalism. CDT defines a (formal) gravitational path integral over geometries (described by all physically distinct metric tensors g)

$$\mathcal{Z}_{\mathrm{QG}} = \int \mathcal{D}_{\mathcal{M}}[g]\, \mathrm{e}^{iS_{\mathrm{HE}[g]}} \quad \rightarrow \quad \mathcal{Z}_{\mathrm{CDT}} = \sum_{\mathcal{T}} \mathrm{e}^{iS_{\mathrm{R}}[\mathcal{T}]} \tag{1}$$

by a sum over lattices (also called *triangulations* $\mathcal{T}$) constructed from four-dimensional simplicial building blocks with fixed lengths of space-like and time-like links. By gluing 4-simplices together, one obtains piecewise linear manifolds where non-trivial curvature is defined by deficit angles depending on how many simplices share common two-dimensional triangles. In CDT, one additionally introduces *causal structure*, consistent with *global hyperbolicity*, which requires that each triangulation admits a global proper time foliation into spatial slices (three-dimensional hypersurfaces) of fixed topology. As changes of spatial topology in time would most likely break causality, one also assumes that the topology of each slice is the same. Such triangulations can be constructed by using two types of 4-simplices, called the $(4,1)$-simplex and the $(3,2)$-simplex[1].

In equation (1), S_{R} is the Hilbert–Einstein action S_{HE} obtained following Regge's method for describing piecewise linear maifolds [3]

$$S_{\mathrm{R}}[\mathcal{T}] = -\left(\kappa_0 + 6\varDelta\right) N_0 + \kappa_4 \left(N_{(4,1)} + N_{(3,2)}\right) + \varDelta N_{(4,1)}\,, \tag{2}$$

where $N_{(4,1)}$, $N_{(3,2)}$ and N_0 denote respectively the number of $(4,1)$-simplices, $(3,2)$-simplices and vertices in a triangulation $\mathcal{T}$. κ_0, $\varDelta$ and κ_4 are three dimensionless bare coupling constants related to Newton's constant, the cosmological constant and the asymmetry between lengths of space-like and time-like links in the triangulation.

In order to study the regularized path integral $\mathcal{Z}_{\mathrm{CDT}}$ (1) in four spacetime dimensions, one is forced to apply the Wick rotation which changes time-like links into space-like links, *i.e.* changes the real (Lorentzian) time $t^{(\mathrm{L})}$ into the imaginary (Euclidean) time $t^{(\mathrm{E})}$ ($t^{(\mathrm{L})} \rightarrow t^{(\mathrm{E})} = -it^{(\mathrm{L})}$), and also changes the Lorentzian action into the Euclidean action ($S_{\mathrm{R}}^{(\mathrm{L})} \rightarrow S_{\mathrm{R}}^{(\mathrm{E})} = -iS_{\mathrm{R}}^{(\mathrm{L})}$). Accordingly, the path integral $\mathcal{Z}_{\mathrm{CDT}}$ becomes a partition function which can be studied numerically using Monte Carlo techniques.

[1] Each 4-simplex has exactly 5 vertices. Due to the imposed proper time foliation, each vertex in the triagulation has a uniquely defined (integer) time coordinate t and the numbers (n,m) denote the number of vertices in t and $t \pm 1$, respectively. For details of the CDT lattice implementation, one can check [4].

Most of the previous studies of four-dimensional CDT were done for the fixed spatial topology of the 3-sphere and time-periodic boundary conditions, such that space-time topology was: $S^3 \times S^1$. In such a case, four distinct phases of quantum geometry, called the A, B, C and C_b phases, were discovered (see figure 1). The most interesting one was the *semiclassical* phase C, where 4-dimensional universe consistent with GR was dynamically emerging from quantum fluctuations [5]. In phase C, the distribution and fluctuations of spatial volume in time were accurately described by the Hartle–Hawking minisuperspace action [6, 7]. In the $S^3 \times S^1$ topology, the A–C transition was classified to be first order, while the B–C_b and the C–C_b transitions were found to be second (or higher) order [8, 9], which in principle may be related with the UV limit of quantum gravity [10, 11].

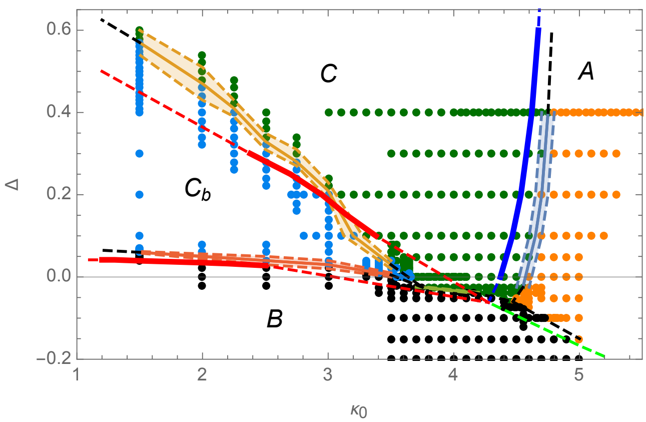

Fig. 1. Phase diagram of the 4-dim CDT with the toroidal topology of spatial slices in the (κ_0, Δ) bare couplings plane (κ_4 is fine-tuned to critical value consistent with infinite volume limit). Points are actual measurements and thin solid lines represent the measured phase transitions (shaded regions are error bars). For comparison, we also plot the phase transitions measured in the spherical spatial topology (thick solid lines).

3. CDT with toroidal spatial topology

The (fixed) topology of spatial slices is a free parameter in CDT. Since we do not know the real topology of the Universe, we can only check if and how the CDT results depend on the spatial topology choice. Below, we present new results obtained for the fixed spatial topology of the 3-torus (and the time periodic boundary conditions, resulting in the space-time topology: $T^3 \times S^1$) which has been studied recently.

In the toroidal spatial topology, one can observe the analogue of the *semiclassical* phase C, but the average spatial volume distribution in time $\langle V_3(t) \rangle$ changes from that of the (Euclidean) de Sitter space in the spherical CDT to the flat profile in the toroidal case. This is not surprising as in principle different topological conditions may favour different solutions of GR and the quantum fluctuations occur around a different semiclassical background geometry in each case. Nevertheless, in the toroidal CDT case, the fluctuations of $V_3(t)$ are again well-described by the minisuperspace action. Now, due to the lack of the semiclassical potential term in the effective action, one can also observe a quantum correction of the potential, which could not be measured in the spherical CDT [12, 13].

In order to identify various phases of quantum geometry, one can define the following order parameters, which were earlier used in the spherical CDT:

$$\begin{aligned} \mathrm{OP}_1 &= N_0/\left(N_{(4,1)}+N_{(2,3)}\right), & \mathrm{OP}_2 &= N_{(3,2)}/N_{(4,1)}, \\ \mathrm{OP}_3 &= \sum_t (V_3(t+1)-V_3(t))^2, & \mathrm{OP}_4 &= \max_v O(v), \end{aligned} \tag{3}$$

where $O(v)$ is the vertex coordination number, *i.e.* the number of simplices sharing a given vertex v. The behaviour of the order parameters in all CDT phases has been summarized in Table I.

TABLE I

Order parameters used in CDT phase transition studies.

	Phase A	Phase B	Phase C	Phase C_b
OP_1	large	small	medium	medium
OP_2	small	small	large	large
OP_3	medium	large	small	medium
OP_4	small	large	small	large

By measuring the order parameters in various points of the CDT phase diagram (see figure 1), one can show that the analogues of all four phases discovered in the spherical topology are also present in the toroidal topology [14]. The precise position of a given phase transition is signaled by a peak of susceptibility

$$\chi_{\mathrm{OP}} \equiv \left\langle \mathrm{OP}^2 \right\rangle - \langle \mathrm{OP} \rangle^2 \tag{4}$$

of an order parameter OP. The phase diagram in figure 1 shows that the toroidal CDT phase transitions are only slightly shifted *versus* the spherical CDT case which most likely results from different finite size effects in each of the two topologies[2]. The studies of CDT with the space-time topology

[2] The minimal possible triangulation of the 3-torus is much larger than the minimal triangulation of the 3-sphere [12].

$T^3 \times S^1$ performed in the most interesting region of the parameter space, near the conjectured "quadruple" point where all four phases were supposed to meet, revealed that in fact there are two separate "triple" points where three of the four phases meet[3]. As a result, one can observe a direct C–B phase transition line (see figure 1).

The order parameters (3) can also be used to measure the order of the phase transitions. By looking at the Monte Carlo history of an order parameter at the transition point, one can check if the parameter jumps between two different states which can be a sign of a first-order transition. If the separation of the states is large enough, one typically observes a hysteresis at the transition region. One should also carefully analyse finite size effects related to the (finite) volume of triangulations fixed in the numerical simulations, *e.g.* the separation of the states can either increase or decrease with the lattice volume which can imply the first or the higher order transition, respectively. Due to the finite size effects, the position of the transition point will also typically shift in the parameter space when the lattice volume is increased and one can measure critical exponents related to this shift. The critical exponent (close to) one suggests a first order transition, while a higher value of the exponent is typical for a higher order transition. By using these tools, one was able to show that the (recently discovered) direct C–B transition is most likely first order [15], albeit with some untypical properties suggesting that the end points can be higher order. In the $T^3 \times S^1$ topology, it was also confirmed that the A–C transition is first order [16] and the B–C_b transition is higher order, exactly as it was observed in the $S^3 \times S^1$ topology. The question mark remains for the C–C_b transition, which was shown to be second (or higher) order in the spherical case. In the toroidal CDT, one observes very strong hysteresis in the transition region, suggesting a first order transition. So far, due to the hysteresis, the numerical algorithms used in the Monte Carlo simulations do not allow for precise finite size scaling analysis of that transition so one can neither prove or disprove this hypothesis.

4. Summary and conclusions

In principle, the CDT results may depend on the choice of (fixed) spatial topology. Most of the previous studies of CDT were done for the spatial topology of the 3-sphere and the time periodic boundary conditions. We have briefly presented the recent results of CDT with the spatial topology of the 3-torus. We have shown that the phase structure and the order of the measured phase transitions have not changed due to the topology change

[3] In the $S^3 \times S^1$ CDT the autocorrelation time measured in the numerical simulations performed in this region of the parameter space was extremely large and triangulations got effectively "freezed" in the Monte Carlo time making precise phase transition studies impossible.

(see figure 1 and Table II). The studies provide evidence that the CDT results are universal, independently of the real topology of the Universe.

TABLE II

Comparison of phase transitions observed in the toroidal and in the spherical CDT.

Phase transition	Topology: $S^1 \times T^3$	Topology: $S^1 \times S^3$
A–C	1st order	1st order
B–C	1st order	?
B–C_b	higher order	higher order
C–C_b	?	higher order

The author acknowledges support of the grant UMO-2016/23/ST2/00289 from the National Science Centre, Poland (NCN). The results described herein were obtained in collaboration with J. Jurkiewicz, J. Ambjørn, A. Görlich, D.N. Coumbe, Z. Drogosz, D. Németh, and G. Czelusta.

REFERENCES

[1] G. 't Hooft, M.J.G. Veltman, *Ann. Inst. Henri Poincaré Phys. Theor. A* **20**, 69 (1974).

[2] S. Weinberg, «Ultraviolet Divergences in Quantum Theories of Gravitation», in: S. Hawking, W. Israel (Eds.) «General Relativity, an Einstein Centenary Survey», *Cambridge University Press*, 1979.

[3] T. Regge, *Nuovo Cim.* **19**, 558 (1961).

[4] J. Ambjørn, J. Jurkiewicz, R. Loll, *Nucl. Phys. B* **610**, 347 (2011).

[5] J. Ambjørn *et al.*, *Phys. Rev. Lett.* **100**, 091304 (2008).

[6] J. Ambjørn, A. Görlich, J. Jurkiewicz, R. Loll, *Phys. Rev. D* **78**, 063544 (2008).

[7] J. Ambjørn *et al.*, *Nucl. Phys. B* **849**, 144 (2011).

[8] J. Ambjørn, S. Jordan, J. Jurkiewicz, R. Loll, *Phys. Rev. D* **85**, 124044 (2012).

[9] J. Ambjørn *et al.*, *Phys. Rev. D* **95**, 124029 (2017).

[10] J. Ambjørn *et al.*, *Class. Quantum Grav.* **31**, 165003 (2014).

[11] J. Ambjørn, D. Coumbe, J. Gizbert-Studnicki, J. Jurkiewicz, *Phys. Rev. D* **93**, 104032 (2016).

[12] J. Ambjørn *et al.*, *Phys. Rev. D* **94**, 044010 (2016).

[13] J. Ambjørn *et al.*, *Nucl. Phys. B* **922**, 226 (2017).

[14] J. Ambjørn *et al.*, *J. High Energy Phys.* **1806**, 111 (2018).

[15] J. Ambjørn *et al.*, *J. High Energy Phys.* **1907**, 166 (2019).

[16] J. Ambjørn *et al.*, *Class. Quantum Grav.* **36**, 224001 (2019).